Phys 410
Spring 2013
Lecture #32 Summary
12 April, 2013

We considered the most general motion of a system of particles. We specifically consider rigid bodies, defined as multi-particle objects in which the distance between any two particles never changes as the object moves. We then considered the kinetic energy of rotation of a rigid body and found that it decomposes cleanly into the kinetic energy of the center of mass (relative to some origin), and the kinetic energy of motion relative to the CM. For a rigid body, the only motion it can have relative to the CM is rotation.

Next we considered an arbitrary rigid object that is forced to rotate about a single fixed axis, which we take to the z-axis. The angular velocity of the object can be written as $\vec{\omega} = \omega\hat{z}$. Naively we might expect that the angular momentum of the object to be $\vec{L} = I_z\vec{\omega}$, where $I_z = \sum_\alpha^N m_\alpha \rho_\alpha^2$ is the moment of inertia for rotation about that axis. This turns out to be true only in special cases of very symmetric objects, or when the axis of rotation is chosen along one of the 'principal axes', defined below. We did the full general calculation of $\vec{L}$ and found that $\vec{L} = L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$, where $L_x = -\sum_\alpha^N m_\alpha x_\alpha z_\alpha\, \omega$, $L_y = -\sum_\alpha^N m_\alpha y_\alpha z_\alpha\, \omega$, and $L_z = -\sum_\alpha^N m_\alpha (x_\alpha^2 + y_\alpha^2)\, \omega$. Thus in general the angular momentum vector $\vec{L}$ is not parallel to the axis of rotation $\hat{\omega}$.

Next we considered an arbitrary rigid body rotating about an arbitrary axis. In general the axis of rotation of an object will change as it moves. We calculated $\vec{L}$ by summing over all particles in the system and found that the vector quantity could be broken down into components as $L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$, with $I_{xx} = \sum_\alpha^N m_\alpha (y_\alpha^2 + z_\alpha^2)$, $I_{xy} = -\sum_\alpha^N m_\alpha x_\alpha y_\alpha$, $I_{xz} = -\sum_\alpha^N m_\alpha x_\alpha z_\alpha$, and similar expressions for L_y and L_z. All of these results can be summarized in a simple matrix equation as $\vec{L} = \bar{\bar{I}}\vec{\omega}$, where $\vec{L} = \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$ is the angular momentum represented as a column vector, $\bar{\bar{I}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$ is called the inertia tensor, and $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$ is the angular velocity vector. Note that the inertia tensor is symmetric about the diagonal: $I_{ij} = I_{ji}$. $\vec{L} = \bar{\bar{I}}\vec{\omega}$ is a general expression relating the angular momentum vector to the axis of rotation.

We did the example of a cube of side a and mas M rotated about one edge. The inertia tensor can be calculated by converting the sums to integrals, for example: $I_{xx} = \sum_{\alpha}^{N} m_{\alpha}(y_{\alpha}^2 + z_{\alpha}^2) \xrightarrow{yields} \int_0^a dx \int_0^a dy \int_0^a dz\ \rho\,(y^2 + z^2)$, where $\rho = M/a^3$ is the density of the uniform cube. Here we assume that the corner of the cube (at the origin of the Cartesian coordinate system) will remain fixed. The resulting inertia tensor for this case is $\bar{\bar{I}} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$. This inertia tensor can be used for any rotation axis that passes through the corner of the cube at the origin. In particular, for rotation about the x-axis, $\vec{\omega} = (\omega, 0,0)$ and we find the angular momentum to be $\vec{L} = Ma^2\omega\left(\frac{2}{3}, -\frac{1}{4}, -\frac{1}{4}\right)$. It is clear in this case that $\vec{L}$ is not parallel to $\vec{\omega}$. This is due in part to the fact that the object is not symmetric with respect to the axis of rotation.

A surprising result is that any object, no matter how irregular, always has 3 principal axes, for which the angular momentum vector and angular velocity are parallel. In other words, for any object we can find three perpendicular axes around which the object will rotate without "wobbling". The formal statement is this: For any rigid body and any point O there are three mutually perpendicular principal axes through O. This amounts to finding three perpendicular axes through O for the calculation of the inertia tensor yields a diagonal matrix. This result arises from the linear algebraic properties of any real symmetric matrix (namely $\bar{\bar{I}}$) – it can always be diagonalized.

How to find the principal axes of an arbitrary object? We are looking for three directions for the angular velocity vector $\vec{\omega}$ to create an angular momentum vector that satisfies $\vec{L} = \lambda\vec{\omega}$, where λ is some real number. This is the condition for two vectors to be parallel. Since in addition we know that in general $\vec{L} = \bar{\bar{I}}\vec{\omega}$, we can combine these two equations to find: $\bar{\bar{I}}\vec{\omega} = \lambda\vec{\omega}$, which is a classic eigenvalue problem. This equation states that a matrix multiplying a vector produces the same vector multiplied by a real number, the eigenvalue. The eigenvectors of this equation constitute the angular velocity directions that diagonalize the intertia tensor, and constitute the principal axes. These three vectors span the 3-dimensional coordinate space and are therefore mutually perpendicular.

We write $\lambda\vec{\omega} = \lambda\bar{\bar{1}}\vec{\omega}$, where $\bar{\bar{1}}$ is the 3x3 unit matrix, and then construct the eigenvalue matrix equation: $\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right)\vec{\omega} = 0$. The only way to get non-trivial solutions from this equation is to make $det\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right) = 0$. This yields three eigenvlaues and three eigen-functions. We examined the case of the cube rotated on an axis that passes through one corner of the cube, for which we calculate the inertia tensor above. This inertia tensor yields a characteristic equation $det\left(\bar{\bar{I}} - \lambda\bar{\bar{1}}\right) = (2\mu - \lambda)(11\mu - \lambda)^2 = 0$, where $\mu = Ma^2/12$, giving $\lambda = 2\mu$ as an eigenvalue and $\lambda = 11\mu$ as a double eigenvalue. The eigenvector associated with $\lambda_1 = 2\mu$ is $\widehat{\omega_1} = \frac{1}{\sqrt{3}}(1,1,1)$, which represents the body diagonal of the cube. The cube has a moment of inertia of $2\mu = Ma^2/6$ for rotation about this axis. The other two eigenvalues yield only the condition $\omega_x + \omega_y + \omega_z = 0$ on the eigenvectors, which simply mean that they have to be perpendicular to $\widehat{\omega_1}$. We are free to choose any two such directions that are mutually perpendicular. A set of possible choices are $\widehat{\omega_2} = \frac{1}{\sqrt{6}}(2,-1,-1)$, and $\widehat{\omega_3} = \frac{1}{\sqrt{2}}(0,1,-1)$, for which the cube has moment of inertia $11\mu = 11Ma^2/12$. To summarize, the principal axes $\widehat{\omega_1}$, $\widehat{\omega_2}$, $\widehat{\omega_3}$ diagonalize the inertia tensor as

$$\bar{\bar{I}} = \frac{Ma^2}{12}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{pmatrix}.$$